

ON SOME CONNECTIONS BETWEEN 2D SYSTEMS
THEORY AND THE THEORY OF SYSTEMS OVER RINGS

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1. INTRODUCTION

The analysis of the dynamics of a 2D system with coefficient over a field K is conveniently developed by resorting to polynomial (or serial) representations of sequences of local states [1,2]. In this way, one time variable is associated to the gradation of a polynomial (or serial) module and the state updating equation of the resulting system has the standard structure of a 1D free linear system over the ring of polynomials [3].

Nevertheless, some problems arising in 2D systems cannot be solved in the general context of a theory of systems over rings. In fact, mainly in realization and control problems, the solutions given by the theory of systems over rings do not necessarily lead to systems having 2D structure, that is to systems where the causality connected to the partial ordering induced in $\mathbb{Z} \times \mathbb{Z}$ and the first order state updating recursion are preserved.

2. 2D RECURSIVE STATE EQUATIONS AND INTERNAL PROPERTIES

The dynamics of a 2D system is represented by the following updating equations [1]:

$$x(h+1, k+1) = A_1 x(h+1, k) + A_2 x(h, k+1) + B_1 u(h+1, k) + B_2 u(h, k+1) \quad , \quad (1)$$

$$y(h, k) = Cx(h, k) \quad .$$

where the local state x is an n -dimensional vector over a field K , input and output values are scalar and A_1, A_2, B_1, B_2, C are matrices of suitable dimensions with entries in K .

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The global states on the separation sets

$$\mathcal{C}_i = \{(h,k) \in \mathbb{Z} \times \mathbb{Z}, h+k=i\}, \quad i=0, \pm 1, \dots$$

are the elements of the direct product of the local state spaces on \mathcal{C}_i . Bilateral Laurent formal power series provide a convenient tool for representing the global state dynamics.

According to this approach, let

$$x_i = \sum_{j=-\infty}^{+\infty} x(i-j, j) \xi^j \quad (2)$$

represent the global state on \mathcal{C}_i , and

$$u_i = \sum_{j=-\infty}^{+\infty} u(i-j, j) \xi^j, \quad y_i = \sum_{j=-\infty}^{+\infty} y(i-j, j) \xi^j \quad (3)$$

the restrictions to \mathcal{C}_i of input and output functions. With this notation, input and output functions can be written as

$$u = \sum_{i=h}^{+\infty} u_i \eta^i, \quad y = \sum_{i=k}^{+\infty} y_i \eta^i \quad (4)$$

where h and k are integers. The set $K_b^m((\xi))$ of (bilateral) Laurent formal power series with coefficients in K^m can be naturally endowed with the structure of a $K[\xi, \xi^{-1}]$ -module, where $K[\xi, \xi^{-1}]$ is the subring of $K(\xi)$ generated by K, ξ and ξ^{-1} . As a consequence of the module structure, the global state updating equations are

$$x_{i+1} = (A_1 + A_2 \xi) x_i + (B_1 + B_2 \xi) u_i, \quad y_i = C x_i \quad (5)$$

If we restrict global states to belong to $K((\xi))^n$ and input functions to have the form

$$\sum_{i=h}^{+\infty} u_i \eta^i, \quad u_i \in K((\xi)),$$

system (5) can be viewed as a linear system over the field $K((\xi))$. Then 1D linear theory applies and reachability and observability conditions correspond to assume

that the matrices

$$\mathcal{R} = [(B_1 + B_2 \xi) (A_1 + A_2 \xi)(B_1 + B_2 \xi) \cdots (A_1 + A_2 \xi)^{n-1} (B_1 + B_2 \xi)]$$

$$\mathcal{O} = \begin{bmatrix} C \\ C(A_1 + A_2 \xi) \\ \vdots \\ C(A_1 + A_2 \xi)^{n-1} \end{bmatrix}$$

have full rank over $K((\xi))$.

In the general case, global states and inputs can have infinitely many non-zero elements in both directions of the separation set, so they are really 'bilateral' and cannot be represented on $K((\xi))$ or $K((\xi^{-1}))$. While global reachability of system (1) and reachability of system (5) over the field $K((\xi))$ are equivalent, global observability of (1) is not implied by observability of (5) over $K((\xi))$, so that global states with unilateral support which are distinguishable from each other can be undistinguishable from global states with bilateral support.

Theorem 1. [2] *The following facts are equivalent:*

- (i) *the 2D system (1) is globally reachable*
- (ii) *$\det \mathcal{R} \neq 0$*
- (iii) *there exists an integer $N(\geq n^2)$ such that any set of local states*

$$x(0,0), \quad x(-1,1), \quad \dots, \quad x(-N+1, N-1)$$

on $\mathcal{S} = \{0,0\}, (-1,1), \dots, (-N+1, N-1)\}$ is the restriction to \mathcal{S} of a global state on \mathcal{C}_0 produced by some input function with compact support on $\mathbb{Z} \times \mathbb{Z}$.

Remark 1. Note that the 2D global reachability condition $\det \mathcal{R} \neq 0$ is weaker than the reachability condition for (5), when considered as a linear system with state space $K[\xi, \xi^{-1}]^n$. The latter condition, which corresponds to $\det \mathcal{R} = k\xi^m (k \neq 0)$, is equivalent to require global states with compact support be reachable by input functions with compact support.

Theorem 2 [2] *The following facts are equivalent:*

- (i) *the 2D system (1) is globally observable*
- (ii) *$\det \mathcal{O} = k\xi^m$ for some integer m and some non zero k in K*
- (iii) *there exists a finite subset $\mathcal{F} \subset \mathbb{Z} \times \mathbb{Z}$ such that $x(0,0)$ is uniquely determined by the free output values in \mathcal{F} , whatever local states may be given on $\mathcal{C}_0 \setminus \{(0,0)\}$.*

Remark 2. If the matrix \mathcal{O} is full rank over $K((\xi))$ and $\det \mathcal{O}$ is not invertible in $K[\xi, \xi^{-1}]$, the subspace of global states which are undistinguishable from zero is finite dimensional over K .

Theorem 3 [4] *System (1) is globally controllable to zero state if and only if $(A_1 + A_2\xi)^n$ factorizes as*

$$(A_1 + A_2\xi)^n = \mathcal{R}M \quad (6)$$

for some rational matrix M in $K(\xi)^{n \times n}$.

Theorem 4 [4] *System (1) is globally reconstructible if and only if $(A_1 + A_2\xi)^n$ factorizes as*

$$(A_1 + A_2\xi)^n = T\mathcal{O} \quad (7)$$

for some polynomial matrix T in $K[\xi, \xi^{-1}]^{n \times n}$.

Remark 3. Global reachability implies global controllability. In fact condition (6) can be fulfilled assuming $M = \mathcal{R}^{-1}(A_1 + A_2\xi)^n$. Also, global observability implies global reconstructibility, since $T = (A_1 + A_2\xi)^n \mathcal{O}^{-1}$ is in $K[\xi, \xi^{-1}]^{n \times n}$ and satisfies (7).

3. DUALITY

Consider the system

$$w(t+1) = F(\xi)w(t) + G(\xi)v(t), \quad z(t) = H(\xi)w(t), \quad (8)$$

defined over the ring of polynomials $K[\xi, \xi^{-1}]$. Here the input set is the ring $K[\xi, \xi^{-1}][n^{-1}]$, the output set is the ring $K[\xi, \xi^{-1}][[n]]$, the states are elements of the free module $K[\xi, \xi^{-1}]^n$ and the matrices $F(\xi)$, $G(\xi)$, $H(\xi)$ have entries in $K[\xi, \xi^{-1}]$.

Denote by \mathcal{R}_p and \mathcal{O}_p its reachability and observability matrices.

Comparing the results from the theory of systems over rings with those summarized in the previous section, we can see that:

(i) the conditions for reachability (observability) of system (8) correspond to the conditions for global observability (global reachability) of a 2D system:

	reachability	observability
system (8)	\mathcal{R}_p unimodular	\mathcal{O}_p full rank
2D system	\mathcal{R} full rank	\mathcal{O} unimodular

(ii) the conditions for controllability (reconstructibility) of system (8) correspond to the conditions for global reconstructibility (global controllability) of a 2D system:

	controllability	reconstructibility
system (8)	$F(\xi)^n = \mathcal{R}_p P$ $P \in K[\xi, \xi^{-1}]^{n \times n}$	$F(\xi)^n = L \mathcal{O}_p$ $L \in K(\xi)^{n \times n}$
2D system	$(A_1 + A_2 \xi)^n = \mathcal{R} M$ $M \in K(\xi)^{n \times n}$	$(A_1 + A_2 \xi)^n = T \mathcal{O}$ $T \in K[\xi, \xi^{-1}]^{n \times n}$

These facts are formally explained by viewing 2D systems as dual of systems over the ring $K[\xi, \xi^{-1}]$.

Let us briefly recall from [2] the main steps in the construction of the dual system of (8).

1. The global state space of the 2D system (1), namely the space $K_b^n((\xi))$, is the algebraic dual of $K^n[\xi, \xi^{-1}]$, which is the state space of system (8):

$$(K^n[\xi, \xi^{-1}])^* = K_b^n((\xi))$$

2. The output space $K_b((\xi))[[n]]$ of (1) is the algebraic dual of the input space $n^{-1}K_b[\xi, \xi^{-1}][n^{-1}]$ of system (8).

Similarly the space of 2D inputs whose support is in $\mathcal{C}_{-n} \dots \mathcal{C}_{-1}$, i.e. whose elements are represented by series in $n^{-1}K_b((\xi))[[n^{-1}]]_n$, is the algebraic dual of the space $K[\xi, \xi^{-1}][n]_n$ of output restrictions to $[0, n-1]$ of system (8):

$$(K[\xi, \xi^{-1}][n]_n)^* = n^{-1}K_b((\xi))[[n^{-1}]]_n$$

Let ρ and ω be the reachability and observability maps of (8):

$$\rho: n^{-1}K[\xi, \xi^{-1}][n^{-1}] \rightarrow K[\xi, \xi^{-1}]^n:$$

$$\sum_{q=1}^Q \sum_p v(-q-p, p) \xi^p n^{-q} \mapsto \sum_{q=1}^Q F(\xi)^{q-1} G(\xi) \sum_p v(-q-p, p) \xi^p,$$

$$\omega: K[\xi, \xi^{-1}]^n \rightarrow K[\xi, \xi^{-1}][[n]]: \sum_i x_i \xi^i \mapsto \sum_{j=0}^{\infty} H(\xi) F(\xi)^j \sum_i x_i \xi^i n^j,$$

and denote by π_n the projection map of $K[\xi, \xi^{-1}][[n]]$ onto the subspace $K[\xi, \xi^{-1}][n]_n$ of polynomials with degree less than n .

In the diagram

$$\begin{array}{ccccc} n^{-1}K[\xi, \xi^{-1}][n^{-1}] & \xrightarrow{\rho} & K[\xi, \xi^{-1}]^n & \xrightarrow{\omega_n} & K[\xi, \xi^{-1}][n]_n \\ & & \searrow \omega & \nearrow \pi_n & \\ & & K[\xi, \xi^{-1}][[n]] & & \end{array}$$

ρ is injective if and only if system (8) is reachable, ω and $\omega_n := \pi_n \circ \omega$ are surjective if and only if system (8) is observable.

In the dual sequence

$$K_b((\xi))[[n]] \xleftarrow{\rho^*} K_b((\xi))^n \xleftarrow{\omega_n^*} n^{-1}K_b((\xi))[[n^{-1}]]_n$$

ρ^* and ω_n^* are the dual maps of ρ and ω_n and provide the observability and the

n-steps reachability maps of the dual system of (8) given by

$$\bar{w}(t+1) = F^T(\xi)\bar{w}(t) + H^T(\xi)\bar{z}(t), \quad \bar{v}(t) = G^T(\xi)\bar{w}(t) \quad (9)$$

where the input and output alphabets are Laurent formal power series and the state space is $K_b((\xi))^n$.

Then, from the theory of dual spaces we have that ρ^* is injective if and only if ρ is surjective and ω_n^* is surjective if and only if ω_n is injective.

Since the reachability map of system (9)

$$\omega': n^{-1}K_b((\xi))[n^{-1}] \rightarrow K_b((\xi))^n$$

is surjective if and only if ω_n^* is surjective, reachability (observability) of system (9) is equivalent to observability (reachability) of system (8). In particular, assuming

$$F(\xi) = A_1^T + A_2^T \xi, \quad G(\xi) = C^T, \quad H(\xi) = B_1^T + B_2^T \xi,$$

we have proved the following theorem:

Theorem 5 [2] *The 2D system (1) is globally reachable (observable) if and only if the system*

$$w(t+1) = (A_1^T + A_2^T \xi)w(t) + C^T v(t), \quad z(t) = (B_1^T + B_2^T \xi)w(t), \quad (10)$$

defined over the ring $K[\xi, \xi^{-1}]$ is observable (reachable).

By projectivity of the module $K[\xi, \xi^{-1}]^n$, the controllability condition of (10),

$$\text{Im}(A_1^T + A_2^T \xi)^n \subseteq \text{Im } \mathcal{R}_p$$

is equivalent to the existence of a $K[\xi, \xi^{-1}]$ -module morphism φ which makes the following diagram

$$\begin{array}{ccc}
 K[\xi, \xi^{-1}]^n & \xrightarrow{(A_1^T + A_2^T \xi)^n} & K[\xi, \xi^{-1}]^n \\
 & \searrow \varphi & \uparrow \mathcal{R}_p \\
 & & K[\xi, \xi^{-1}]^n \cong \eta^{-1} K[\xi, \xi^{-1}] [\eta^{-1}]_n
 \end{array} \quad (11)$$

commutative.

On the other side by the injectivity of the $K[\xi, \xi^{-1}]$ -module $K_b^n((\xi))$ the reconstructibility condition of (1),

$$\ker(A_1 + A_2 \xi)^n \supseteq \ker \mathcal{O} \quad (12)$$

is equivalent to the existence of a $K[\xi, \xi^{-1}]$ -module morphism ψ which makes the following diagram

$$\begin{array}{ccc}
 K_b^n((\xi)) & \xleftarrow{(A_1 + A_2 \xi)^n} & K_b^n((\xi)) \\
 & \searrow \psi & \downarrow \mathcal{O} \\
 & & K_b((\xi))[\eta]_n \cong K_b^n((\xi))
 \end{array} \quad (13)$$

commutative.

Theorem 6 Global reconstructibility of the 2D system (1) is equivalent to controllability of the system (10) defined over $K[\xi, \xi^{-1}]$.

Proof. Assume first commutativity of the diagram (11). Since each of its maps admits a dual map, we have

$$(A_1^T + A_2^T \xi)^n = \varphi^* \mathcal{R}_p^*$$

$$(A_1 + A_2 \xi)^n = \varphi^* \mathcal{O}$$

which guarantees the commutativity of (13) with $\psi = \varphi^*$.

Conversely, assume reconstructibility of (1), i.e. the existence of ψ which makes (13) commutative. Then, by taking the orthogonal complements of (12),

$$\ker(A_1 + A_2 \xi)^{n\perp} \subseteq (\ker \mathcal{O})^\perp$$

and

$$(\ker(A_1^T + A_2^T \xi)^{n*})^\perp \subseteq (\ker \mathcal{R}_p^*)^\perp$$

Hence by the properties of the spaces of linear functionals we have

$$\operatorname{Im}(A_1^T + A_2^T \xi)^n \subseteq \operatorname{Im} \mathcal{R}_p$$

Then there exists φ which makes (11) commutative, and in (13) ψ can be assumed as φ^* .

The reconstructibility condition of system (10)

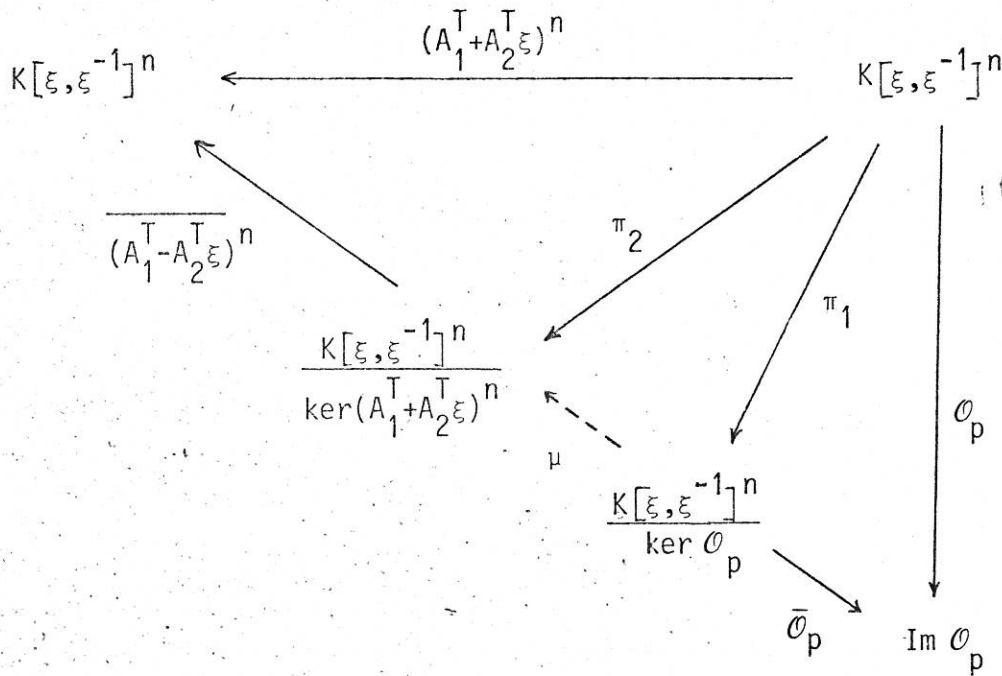
$$\ker(A_1^T + A_2^T \xi)^n \supseteq \ker \mathcal{O}_p \quad (14)$$

is equivalent to the existence of a $K[\xi, \xi^{-1}]$ -module morphism χ which makes the following diagram

$$\begin{array}{ccc} K[\xi, \xi^{-1}]^n & \xleftarrow{(A_1^T + A_2^T \xi)^n} & K[\xi, \xi^{-1}]^n \\ & \nwarrow \chi & \downarrow \mathcal{O}_p \\ & & \operatorname{Im} \mathcal{O}_p \end{array} \quad (15)$$

commutative.

In fact, (14) is an obvious consequence of the existence of χ . Conversely (14) implies the existence of a $K[\xi, \xi^{-1}]$ -module morphism μ which makes the following diagram



commutative, and we can assume $\chi = (A_1^T + A_2^T \xi)^n \cdot \mu \cdot \bar{\mathcal{O}}_p$.

The global controllability condition of system (1)

$$\text{Im}(A_1 + A_2 \xi)^n \subseteq \text{Im } \mathcal{R} \quad (16)$$

is equivalent to the existence of a $K[\xi, \xi^{-1}]$ -module morphism v which makes the following diagram

$$\begin{array}{ccc} K_b^n((\xi)) & \xrightarrow{(A_1 + A_2 \xi)^n} & K_b^n((\xi)) \\ & \searrow v & \uparrow \bar{\mathcal{R}} \\ & & K_b^n((\xi)) / \ker \mathcal{R} \end{array} \quad (17)$$

commutative.

In fact (16) is an easy consequence of the existence of v . Viceversa, assuming $\bar{\mathcal{R}}^{-1}$ as the inverse of $\bar{\mathcal{R}}$ on $\text{Im } \bar{\mathcal{R}}$, the inclusion (16) allows to define $v = \bar{\mathcal{R}}^{-1} \circ (A_1 + A_2 \xi)^n$ which makes the diagram (17) commutative.

Theorem 7 *Global controllability of the 2D system (1) is equivalent to reconstructibility of the system (10).*

Proof. By the same arguments used in Theorem 6, the proof of the equivalence reduces to show that diagrams (15) and (17) are dual.

First we prove that $K_b^n((\xi))/\ker \mathcal{R}$ can be viewed as the algebraic dual of $\text{Im } \mathcal{O}_p$.

Let s be any element in $K_b^n((\xi))$ and denote by $[s]$ its equivalence class modulo $\ker \mathcal{R}$. Then for any q in $K[\xi, \xi^{-1}]^n$, the relation (*)

$$\langle \mathcal{O}_p q, [s] \rangle = (q^T \mathcal{O}_p^T s, \xi^0)$$

defines a linear functional on $\mathcal{O}_p K[\xi, \xi^{-1}]^n$. Viceversa, given a linear functional $f: \mathcal{O}_p K[\xi, \xi^{-1}]^n \rightarrow K$, there exists a bilateral power series s in $K_b^n((\xi))$ such that

$$f(\mathcal{O}_p q) = \langle \mathcal{O}_p q, [s] \rangle$$

for any q in $K[\xi, \xi^{-1}]^n$, and $[s]$ is uniquely determined.

Assume that the map χ in (15) exists, and consider an irreducible matrix fraction representation of it given by NQ^{-1} . Then $NQ^{-1}\mathcal{O}_p$ is a polynomial matrix and \mathcal{O}_p factorizes as

$$\mathcal{O}_p = QH \quad (18)$$

for some H in $K[\xi, \xi^{-1}]^{n \times n}$. (18) follows from the Bézout identity $AN+BQ=I_n$ by pre multiplication by Q and postmultiplication by $Q^{-1}\mathcal{O}_p$.

For any s in $K_b^n((\xi))$, the bilateral series g which solve the equation

$$N^T s = Q^T g \quad (19)$$

are equivalent modulo $\ker \mathcal{R}$, and the map

$$\nu: K_b^n((\xi)) \rightarrow K_b^n((\xi))/\ker \mathcal{R} : s \mapsto g$$

is a well defined $K[\xi, \xi^{-1}]$ -module morphism. ν is the dual map of χ . In fact

(*) As commonly used in formal power series theory, (s, ξ^i) denotes the coefficient of ξ^i in the series s .

$$\begin{aligned}\langle x \otimes_p q, s \rangle &= (q^T \otimes_p^T (Q^{-1})^T N^T s, \xi^0) \\ &= (q^T H^T Q^T (Q^{-1})^T N^T s, \xi^0) = (q^T H^T N^T s, \xi^0)\end{aligned}$$

is equal to

$$\begin{aligned}\langle \otimes_p q, v[s] \rangle &= \langle \otimes_p q, [g] \rangle = (q^T \otimes_p^T g, \xi^0) \\ &= (q^T H^T Q^T g, \xi^0) = (q^T H^T N^T s, \xi^0)\end{aligned}$$

for any s in $K_D^n((\xi))$ and q in $K[\xi, \xi^{-1}]^n$.

4. STATE FEEDBACK STABILIZABILITY

The structure of partial ordering which underlies 2D systems, makes possible to consider state-feedback schemes which cannot be derived by simply extending 1D concepts. It is clear that, following 1D philosophy, we can adopt inputs which depend only on the states at the same "time" and/or inputs whose dynamic dependence on the states fits with the causality induced by the partial ordering in $\mathbb{Z} \times \mathbb{Z}$. So, the resulting systems still keep the state updating structure of a 2D system.

However, as it will formally stated later, it is possible to take into account inputs which depend dynamically on the states but do not respect the causality induced by the partial ordering of $\mathbb{Z} \times \mathbb{Z}$. So doing, in general the 2D structure of the original system is destroyed and so called [5] "weakly causal" 2D systems are obtained.

In the following we shall give an insight on the application of different state-feedback schemes to the stabilization of 2D systems given on the real field.

Referring to the state feedback structures, we can essentially deal with the following two situations

a) *state feedback preserving 2D causality*

a₁) static:

$$u(h, k) = Kx(h, k), \quad K \in \mathbb{R}^{1 \times n} \quad (20)$$

a₂) 2D recursive:

$$u(h,k) = \sum_{i+j=1}^q \alpha_{ij} u(h-i,k-j) + \sum_{i,j=0}^p K_{ij} x(h-i,k-j), \quad (21)$$

$$\alpha_{ij} \in \mathbb{R}, \quad K_{ij} \in \mathbb{R}^{1 \times n}$$

b) *state feedback producing 2D weakly causal systems:*

$$u(h,k) = \sum_{i=-m}^m K_i x(h-i,k+i) \quad K_i \in \mathbb{R}^{1 \times n} \quad (22)$$

or, in formal power series notation,

$$\mathcal{U}_t(\xi) = \left(\sum_{i=-m}^m K_i \xi^i \right) \mathcal{X}_t(\xi) = K(\xi) \mathcal{X}_t(\xi),$$

$$K(\xi) \in \mathbb{R}[\xi, \xi^{-1}]^{1 \times n}$$

Remark 4. State feedback (b) does not preserve the structure of the state updating equation given by (1). In fact the computation of the local state at some point (h,k) belonging to the separation set \mathcal{C}_t involves not only data at points of \mathcal{C}_{t-1} which are less than (h,k) in the partial ordering of $\mathbb{Z} \times \mathbb{Z}$, but also data at points of \mathcal{C}_{t-1} which are not causally related to (h,k) .

Consider a 2D system (A_1, A_2, B_1, B_2, C) and assume $u(h,k) = Kx(h,k)$ (static state feedback). We obtain a new 2D system $(\bar{A}_1, \bar{A}_2, B_1, B_2, C)$ where

$$\bar{A}_1 = A_1 + B_1 K, \quad \bar{A}_2 = A_2 + B_2 K$$

Recalling that the $(\bar{A}_1, \bar{A}_2, B_1, B_2, C)$ is internally stable if and only if the polynomial $\det(I - \bar{A}_1 z_1 - \bar{A}_2 z_2)$ is devoid of zeros in the closed polydisc $|z_1| \leq 1, |z_2| \leq 1$ [1,6], it is straightforward to see that if the 2D system (A_1, A_2, B_1, B_2, C) is stabilizable by means of static state feedback, then the pairs (A_1, B_1) and (A_2, B_2) are

simultaneously stabilizable ($\hat{=}$).

As shown in the following example, the converse of the result above does not hold.

Example. Consider the following 2D system

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -3/4 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3/4 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Since the polynomial

$$\begin{aligned} p(z_1, z_2) &= \det(I - (A_1 + B_1 K)z_1 - (A_2 + B_2 K)z_2) \\ &= 1 + \frac{3}{4}z_1 - \frac{3}{4}z_2 + k_0(z_1 + z_2)^3 + k_1(z_1 + z_2)^2 + k_2(z_1 + z_2) \end{aligned}$$

vanishes in $(-2/3, 2/3)$ for any $K = [k_0 \ k_1 \ k_2]$, (A_1, A_2, B_1, B_2) is not stabilizable by state feedback. Nevertheless (A_1, B_1) and (A_2, B_2) are simultaneously stabilizable.

Consider again a 2D system and assume a state feedback having structure (22), i.e.

$$u_t(\xi) = K(\xi) x_t(\xi)$$

The global state evolves accordingly to the following equation

$$x_{t+1} = [(A_1 + A_2 \xi) + (B_1 + B_2 \xi)K(\xi)] x_t$$

($\hat{=}$) The idea of connecting 2D stability and simultaneous stabilization problems raised in some discussions with C. Byrnes.

It is interesting to examine the following two cases

- (i) $\det \mathcal{R} = k\xi^m$, $k \neq 0$. This condition corresponds to assume that

$$w(t+1) = (A_1 + A_2\xi)w(t) + (B_1 + B_2\xi)v(t) \quad (23)$$

is reachable as system over the ring $\mathbb{R}[\xi, \xi^{-1}]$. Consequently the polynomial

$$\det(\eta I - A_1 - A_2\xi + (B_1 + B_2\xi)K(\xi)) \quad (24)$$

in the indeterminate η is coefficient assignable and hence the 2D system is stabilizable.

Remark 5. It is important to point out that in this case the stabilization can be achieved even when the pairs (A_1, B_1) and (A_2, B_2) are not simultaneously stabilizable as we can easily derive from the following system

$$(A_1, A_2, B_1, B_2, C) = (2, 3, 1, 0, -)$$

- (ii) $\det \mathcal{R} \neq k\xi^m$. This condition corresponds to assume that the 2D system (1) is globally reachable but (5) is not reachable as a system over $\mathbb{R}[\xi, \xi^{-1}]$. This implies that the coefficients of the polynomial (24) cannot be arbitrarily assigned. However this type of state feedback allows us to stabilize 2D systems in cases when static feedback does not give positive results. Actually the following example shows that this type of state feedback can solve the stabilization problem even in cases when the pairs (A_1, B_1) and (A_2, B_2) are not simultaneously stabilizable.

Example. Consider the 2D system $(A_1, A_2, B_1, B_2, -) = (-4.4, 0, 1, \frac{1}{3}, -)$. The reachability matrix $\mathcal{R} = B_1 + B_2\xi = 1 + \frac{1}{3}\xi$ is not unimodular and the pairs (A_1, B_1) and (A_2, B_2) are not simultaneously stabilizable.

Assuming $K(\xi) = 4.8 - 1.2\xi$, the characteristic polynomial

$$(\eta - A_1 - A_2\xi - (B_1 + B_2\xi)K(\xi)) = \eta - 0.4 - 0.4\xi + 0.4\xi^2$$

shows that the free evolution of the global state asymptotically converges to zero.

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